

Hydrodynamic Approach to the Free Electron Laser Instability

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Starting from the Vlasov-Maxwell system, an exact relativistic hydrodynamic closure for a special type water bag distributions satisfying the Vlasov equation has been derived. It has been shown that the hydrodynamic equations are fully equivalent to the original Vlasov-Maxwell equations. The linear stability of the exact hydrodynamic closure has been studied as well. It has been shown that all basic features of the small signal gain can be derived from the fluid dynamic description developed here. Satisfactory agreement with previously reported results has been also found.

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I. INTRODUCTION

Conventional lasers are ubiquitous sources of coherent electromagnetic radiation over a wide range of the spectrum, from the infrared (around 10 mm) down to the ultraviolet (around 200 nm). However, in the far-infrared, the so-called terahertz part of the spectrum (in the range from 30 mm to 1 mm), or at X-ray wavelengths (less than 10 nm), traditional lasers are not easily achievable. There is therefore ever growing interest in alternate sources of intense, ultra fast, coherent radiation covering these two portions of the electromagnetic spectrum.

The free electron laser^{1,2} (FEL) is a relatively new kind of laser, in which the electrons are not bound in atomic or molecular systems, neither they are confined by a lattice. The source of these free electrons is an electron beam accelerated to suitable energy in a linac or a synchrotron. Generally speaking, the free electron laser converts part of the kinetic energy of the electrons into coherent electromagnetic radiation. This conversion is facilitated by a static magnetic field produced by a magnetic device called an undulator³. Free electron lasers can produce continuous and widely tunable coherent radiation in any part of the electromagnetic spectrum. In addition, very short pulses can be emitted at any wavelength with the laser intensity being very high.

A number of theoretical models have been developed over the years in order to describe the free electron laser. Although the first analyses involved techniques from quantum mechanics⁴ and quantum electrodynamics^{5,6}, classical methods have proved to provide a clear and accurate picture of the underlying processes^{7–10}. The self consistent Maxwell-Boltzmann equations have been further transformed into quasi-Bloch equations in order to

stress on the laser physics perspective of the problem¹¹. The plasma physics nature of the electron dynamics moving in the undulator field has been pursued in the kinetic description based on the coupled Vlasov-Maxwell equations; an approach that has been developed by Davidson and coworkers^{12,13}. Many features of the dynamics of free electron lasers can be understood using a simple pendulum model. It comes about because the electromagnetic wave of the radiated field and the magnetic field of the undulator act in tandem on the electron to produce a sinusoidal potential similar to that of a pendulum^{14–16}. This model is to a large extent similar to the so-called single-wave model, widely used in conventional plasma physics¹⁷.

A relativistic cold-fluid dynamics model of an electron beam with uniform cross section propagating axially through a constant-amplitude helical wiggler magnetic field has been developed as well^{18,19}. The present paper can be considered as a further extension in this direction.

Starting from very basic principles of propagation of a relativistic electron in the static magnetic field of an undulator, and assuming that its dynamics depends on the longitudinal coordinate and the time only, we arrive at a coupled Vlasov-Maxwell system of equations in one spatial dimension and time. Taking into account an exact solution of the Vlasov equation in the form of a relativistic water bag distribution, we perform in Section III a reduction of the Vlasov-Maxwell system to an exact closure of relativistic fluid dynamic equations coupled with the wave equations for the radiation field. An interesting feature of the hydrodynamic system thus derived is the fact that the ponderomotive potential together with the pressure law enter the picture in the form of an effective enthalpy. In Section IV, we present the linear stability analysis in the derived fluid dynamic framework, which is fully equivalent to the kinetic description in terms of the self-consistent Vlasov-Maxwell equations. The numerical solution of the dispersion equation derived in Section IV is presented in Section V and certain useful properties of the imaginary part of its roots (the so-called small-signal gain) are discussed. Finally, in Section VI, we draw some

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conclusions.

II. GENERALITIES AND FIRST PRINCIPLES

The present analysis assumes a relativistic electron beam propagating in longitudinal s direction through a helical wiggler magnetic field described by

$$\mathbf{B}_w = B_0(\mathbf{e}_x \cos k_0 s + \mathbf{e}_y \sin k_0 s), \quad (1)$$

where $B_0 = \text{const}$ is the field amplitude, $\lambda_0 = 2\pi/k_0$ is the wavelength, and \mathbf{e}_x and \mathbf{e}_y are unit Cartesian vectors in the plane perpendicular to the propagation direction. The vector potential associated with the wiggler field (1) is given by

$$\mathbf{A}_w = -\frac{B_0}{k_0}(\mathbf{e}_x \cos k_0 s + \mathbf{e}_y \sin k_0 s). \quad (2)$$

In what follows we assume that spatial variations are one-dimensional in nature, so that $\partial/\partial x = \partial/\partial y = 0$ and $\partial/\partial s$ is generally nonzero.

The Hamiltonian of an electron moving in the wiggler field, as well as in the self-consistent electromagnetic field can be written as

$$\mathcal{H} = m\gamma c^2 = c\sqrt{m^2 c^2 + (\mathbf{P} + e\mathbf{A})^2} - e\Phi, \quad (3)$$

where m is the electron rest mass, γ is the relativistic Lorentz factor, c is the velocity of light in vacuo and \mathbf{P} is the canonical momentum. In addition, Φ is the scalar self-potential and \mathbf{A} is the total vector potential $\mathbf{A} = \mathbf{A}_w + \mathbf{A}_r$, where \mathbf{A}_r describes the radiation self-field. It is convenient to pass to dimensionless momenta and new "time", dimensionless Hamiltonian, and electromagnetic potentials

$$\mathbf{p} = \frac{\mathbf{P}}{mc}, \quad \tau = ct, \quad (4)$$

$$H = \gamma = \sqrt{1 + (\mathbf{p} + \mathbf{a})^2} - \varphi, \quad (5)$$

$$\mathbf{a} = \frac{e\mathbf{A}}{mc}, \quad \varphi = \frac{e\Phi}{mc^2}. \quad (6)$$

In order to eliminate the longitudinal component of the vector potential a_s under the square root, we redefine the longitudinal component of the particle momentum by means of a canonical transformation specified by the generating function

$$F_2(\mathbf{x}, \tilde{\mathbf{p}}; \tau) = x\tilde{p}_x + y\tilde{p}_y + s\tilde{p}_s - \int ds a_s(s; \tau). \quad (7)$$

Dropping the tilde in what follows, we write the new Hamiltonian

$$\gamma = \sqrt{1 + p_s^2 + (\mathbf{p} + \mathbf{a})_\perp^2} - \varphi - \frac{\partial}{\partial \tau} \int ds a_s(s; \tau). \quad (8)$$

Here the subscript " \perp " corresponds to the transverse components of the canonical coordinates and fields.

Hamilton's equations of motion can be written as

$$\frac{d\mathbf{x}_\perp}{d\tau} = \frac{\mathbf{p}_\perp + \mathbf{a}_\perp}{\gamma_c}, \quad \frac{d\mathbf{p}_\perp}{d\tau} = 0, \quad (9)$$

$$\frac{ds}{d\tau} = \frac{p_s}{\gamma_c}, \quad \frac{dp_s}{d\tau} = -\frac{\partial \gamma_c}{\partial s} + \mathcal{F}, \quad (10)$$

where γ_c denotes the kinetic term (the square root) in the total Hamiltonian (8). In addition, \mathcal{F} is the electric force

$$\mathcal{F} = \frac{\partial \varphi}{\partial s} + \frac{\partial a_s}{\partial \tau}, \quad (11)$$

acting on the particle. In the present geometry, there are two exact single-particle invariants in the combined external \mathbf{A}_w and self-fields \mathbf{A}_r and Φ configuration. These are the canonical momenta \mathbf{p}_\perp , transverse to the beam propagation direction.

The nonlinear Vlasov equation for the distribution function $f(\mathbf{x}, \mathbf{p}; \tau)$ can be written as

$$\frac{\partial f}{\partial \tau} + \frac{\mathbf{p}_\perp + \mathbf{a}_\perp}{\gamma_c} \cdot \frac{\partial f}{\partial \mathbf{x}_\perp} + \frac{p_s}{\gamma_c} \frac{\partial f}{\partial s} + \left(\mathcal{F} - \frac{\partial \gamma_c}{\partial s} \right) \frac{\partial f}{\partial p_s} = 0. \quad (12)$$

It possesses an exact solution of the form

$$f(\mathbf{x}, \mathbf{p}; \tau) = \delta(p_x) \delta(p_y) F(s, p_s; \tau), \quad (13)$$

where p_x and p_y are the exact invariants, defined by the last two of Eqs. (9). Expression (13) implies that the transverse motion of the electrons is "cold" since the transverse beam emittance has been neglected. The evolution of the yet unknown function $F(s, p_s; \tau)$ of the longitudinal canonical coordinates and time is governed by the one-dimensional Vlasov equation

$$\frac{\partial F}{\partial \tau} + \frac{p_s}{\gamma_c} \frac{\partial F}{\partial s} + \left(\mathcal{F} - \frac{\partial \gamma_c}{\partial s} \right) \frac{\partial F}{\partial p_s} = 0, \quad (14)$$

where γ_c defined as

$$\gamma_c(s, p_s; \tau) = \sqrt{1 + p_s^2 + a^2(s; \tau)}, \quad (15)$$

is the dimensionless kinetic energy for $p_x = p_y = 0$ with $a^2 = a_x^2 + a_y^2$.

III. DESCRIPTION OF ONE-DIMENSIONAL BEAM PROPAGATION FOR A UNIFORM PHASE-SPACE DENSITY

It can be verified that there exist a class of exact solutions to the one-dimensional Vlasov equation (14) of the form^{20,21}

$$F(s, p_s; \tau) = \mathcal{C} \left\{ \Theta \left[p_s - p_s^{(-)}(s; \tau) \right] - \Theta \left[p_s - p_s^{(+)}(s; \tau) \right] \right\}, \quad (16)$$

where $\Theta(z)$ is the well-known Heaviside step function. Expression (16) implies that the distribution function has a constant phase-space density defined by the constant \mathcal{C} within a simply connected region confined by the curves $p_s^{(-)}(s; \tau)$ and $p_s^{(+)}(s; \tau)$ and is zero outside. If $F(s, p_s; \tau)$ satisfies Eq. (16) initially at $\tau = 0$, then the nonlinear Vlasov equation (14) assures that the phase-space density remains constant²¹ at subsequent values of τ as the boundary curves $p_s^{(-)}(s; \tau)$ and $p_s^{(+)}(s; \tau)$ distort and evolve nonlinearly in response to the applied external and the self-generated fields. Clearly, the area number density

$$N_b = \int ds dp_s F(s, p_s; \tau), \quad (17)$$

is preserved.

Next, we derive the evolution equations for the boundary curves $p_s^{(-)}(s; \tau)$ and $p_s^{(+)}(s; \tau)$. Substituting the explicit form of the solution (16) into the Vlasov equation (14), we obtain

$$\begin{aligned} & -\delta(p_s - p_s^{(-)}) \frac{\partial p_s^{(-)}}{\partial \tau} + \delta(p_s - p_s^{(+)}) \frac{\partial p_s^{(+)}}{\partial \tau} \\ & + \frac{p_s}{\gamma_c} \left[-\delta(p_s - p_s^{(-)}) \frac{\partial p_s^{(-)}}{\partial s} + \delta(p_s - p_s^{(+)}) \frac{\partial p_s^{(+)}}{\partial s} \right] \\ & + \frac{\partial \gamma_c}{\partial s} \left[-\delta(p_s - p_s^{(-)}) + \delta(p_s - p_s^{(+)}) \right] \\ & = \mathcal{F} \left[-\delta(p_s - p_s^{(-)}) + \delta(p_s - p_s^{(+)}) \right]. \end{aligned} \quad (18)$$

Multiplying the above equation by 1, p_s and p_s^2 and integrating over p_s , we find the evolution equations for the boundary curves in the form

$$\frac{\partial}{\partial \tau} (p_s^{(+)} - p_s^{(-)}) + \frac{\partial}{\partial s} (\gamma_c^{(+)} - \gamma_c^{(-)}) = 0, \quad (19)$$

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial \tau} (p_s^{(+2)} - p_s^{(-2)}) + p_s^{(+)} \frac{\partial \gamma_c^{(+)}}{\partial s} - p_s^{(-)} \frac{\partial \gamma_c^{(-)}}{\partial s} \\ & = (p_s^{(+)} - p_s^{(-)}) \mathcal{F}, \end{aligned} \quad (20)$$

$$\begin{aligned} & \frac{1}{3} \frac{\partial}{\partial \tau} (p_s^{(+3)} - p_s^{(-3)}) + p_s^{(+2)} \frac{\partial \gamma_c^{(+)}}{\partial s} - p_s^{(-2)} \frac{\partial \gamma_c^{(-)}}{\partial s} \\ & = (p_s^{(+2)} - p_s^{(-2)}) \mathcal{F}, \end{aligned} \quad (21)$$

where in accordance with Eq. (15), we have defined

$$\gamma_c^{(\pm)}(s; \tau) = \sqrt{1 + p_s^{(\pm)}(s; \tau)^2 + a^2(s; \tau)}. \quad (22)$$

Let us now introduce the hydrodynamic variables n , V and Γ as

$$\mathcal{C} (p_s^{(+)} - p_s^{(-)}) = n\Gamma, \quad \mathcal{C} (\gamma_c^{(+)} - \gamma_c^{(-)}) = nV\Gamma, \quad (23)$$

$$\frac{\mathcal{C}}{2} (p_s^{(+2)} - p_s^{(-2)}) = \frac{\mathcal{C}}{2} (\gamma_c^{(+2)} - \gamma_c^{(-2)}) = nV\Gamma^2. \quad (24)$$

Here $n(s; \tau)$ is the number density, $V(s; \tau)$ is the macroscopic flow velocity and $\Gamma(s; \tau)$ is yet unknown function to be specified later. From Eqs. (23) and (24), we readily obtain

$$p_s^{(+)} = \Gamma \left(V + \frac{n}{2\mathcal{C}} \right), \quad p_s^{(-)} = \Gamma \left(V - \frac{n}{2\mathcal{C}} \right), \quad (25)$$

$$\gamma_c^{(+)} = \Gamma \left(1 + \frac{nV}{2\mathcal{C}} \right), \quad \gamma_c^{(-)} = \Gamma \left(1 - \frac{nV}{2\mathcal{C}} \right). \quad (26)$$

Note from the above equations that the flow velocity V and the function Γ are given by

$$\Gamma V = \frac{1}{2} (p_s^{(+)} + p_s^{(-)}), \quad \Gamma = \frac{1}{2} (\gamma_c^{(+)} + \gamma_c^{(-)}). \quad (27)$$

From the second of Eqs. (27) with due account of Eqs. (25) and (26), we can express the function Γ as

$$\Gamma = \sqrt{\frac{1 + a^2}{(1 - V^2)(1 - 2v_T^2 n^2)}}, \quad (28)$$

where

$$v_T^2 = \frac{1}{8\mathcal{C}^2}, \quad (29)$$

is the thermal speed squared.

To complete the macroscopic fluid description, we need to express the source terms entering the corresponding wave equations for the electromagnetic potentials as functions of n , V and Γ . In the Lorentz gauge

$$\frac{\partial a_s}{\partial s} + \frac{\partial \varphi}{\partial \tau} = 0, \quad (30)$$

the scalar potential φ and the vector potential \mathbf{a} satisfy the wave equations

$$\square \varphi = \frac{e^2}{\epsilon_0 m c^2} \int dp_s F(s, p_s; \tau), \quad (31)$$

$$\begin{aligned} \square \mathbf{a}_\perp &= \frac{\mu_0 e^2}{m} \mathbf{a}_\perp \int \frac{dp_s}{\gamma_c} F(s, p_s; \tau) \\ &+ \frac{\omega_c k_0}{c} (\mathbf{e}_x \cos k_0 s + \mathbf{e}_y \sin k_0 s), \end{aligned} \quad (32)$$

$$\square a_s = \frac{\mu_0 e^2}{m} \int dp_s \frac{p_s}{\gamma_c} F(s, p_s; \tau), \quad (33)$$

where $\square = \partial_s^2 - \partial_\tau^2$ is the well-known d'Alembert operator and

$$\omega_c = \frac{eB_0}{m}, \quad (34)$$

is the electron cyclotron frequency associated with the amplitude of the wiggler field. The integral on the right-hand-side of Eq. (32) can be expressed as

$$\begin{aligned} \int \frac{dp_s}{\gamma_c} F(s, p_s; \tau) &= C \ln \left(\frac{p_s^{(+)} + \gamma_c^{(+)}}{p_s^{(-)} + \gamma_c^{(-)}} \right) \\ &= C \ln \left(1 + \frac{n}{2C} \right) - C \ln \left(1 - \frac{n}{2C} \right), \end{aligned} \quad (35)$$

while the integrals on the right-hand-side of Eqs. (31) and (33) are standard and yield simply $n\Gamma$ and $nV\Gamma$, respectively.

Expressing Eqs. (19) and (20) for the moments in terms of the hydrodynamic variables, we have

$$\frac{\partial}{\partial \tau}(n\Gamma) + \frac{\partial}{\partial s}(nV\Gamma) = 0, \quad (36)$$

$$\frac{\partial}{\partial \tau}(V\Gamma) + \frac{\partial \Gamma}{\partial s} = \mathcal{F}, \quad (37)$$

and from Eq. (21) we obtain yet a third equation

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(n\Gamma^3 V^2 + \frac{n^3 \Gamma^3}{12C^2} \right) + \Gamma^2 V^2 \frac{\partial}{\partial s}(nV\Gamma) \\ + 2n\Gamma^2 V \frac{\partial \Gamma}{\partial s} + \frac{n^2 \Gamma^2}{4C^2} \frac{\partial}{\partial s}(nV\Gamma) = 2n\Gamma^2 V \mathcal{F}, \end{aligned} \quad (38)$$

which is a direct consequence of the first two Eqs. (36) and (37). The latter implies that the hierarchy of macroscopic fluid equations is closed and Eqs. (36) and (37) comprise a complete hydrodynamic closure, fully equivalent to the one-dimensional Vlasov equation (14).

The macroscopic fluid equations (36) and (37) must be supplemented with the equations for the self-fields. The wave equations for the scalar potential φ and for the longitudinal component of the vector potential a_s can be written in a straightforward manner to give

$$\square \varphi = \frac{e^2}{\epsilon_0 m c^2} n\Gamma, \quad (39)$$

$$\square a_s = \frac{\mu_0 e^2}{m} nV\Gamma. \quad (40)$$

Note that the integral in Eq. (35) can be expressed as

$$\int \frac{dp_s}{\gamma_c} F(s, p_s; \tau) = n \left(1 + \frac{2}{3} v_T^2 n^2 \right) + O\left(\frac{1}{C^4}\right). \quad (41)$$

Standard procedure is to formally apply the thermodynamic limit in which C grows infinitely together with the

volume occupied by the electron beam such that the number density n remains finite. In this approximation the thermal speed v_T becomes infinitely small as well. To second order, we can rewrite Eq. (32) as

$$\begin{aligned} \square \mathbf{a}_\perp &= \frac{\mu_0 e^2 n}{m} \left(1 + \frac{2}{3} v_T^2 n^2 \right) \mathbf{a}_\perp \\ &+ \frac{\omega_c k_0}{c} (\mathbf{e}_x \cos k_0 s + \mathbf{e}_y \sin k_0 s). \end{aligned} \quad (42)$$

In order to eliminate the explicit dependence of the right-hand-side of Eq. (42) on the longitudinal coordinate s , we introduce the helical field variables according to

$$\mathcal{A}_x = a_x \cos k_0 s + a_y \sin k_0 s, \quad (43)$$

$$\mathcal{A}_y = -a_x \sin k_0 s + a_y \cos k_0 s. \quad (44)$$

Note that $a^2 = a_x^2 + a_y^2 = \mathcal{A}_x^2 + \mathcal{A}_y^2$. Finally, we collect the macroscopic fluid equations (36) and (37) together with the equations for the electromagnetic fields to write the basic system

$$\frac{\partial}{\partial \tau}(n\Gamma) + \frac{\partial}{\partial s}(nV\Gamma) = 0, \quad (45)$$

$$\frac{\partial}{\partial \tau}(V\Gamma) + \frac{\partial \Gamma}{\partial s} = \mathcal{F}, \quad (46)$$

$$\square \varphi = \frac{e^2}{\epsilon_0 m c^2} n\Gamma, \quad (47)$$

$$\square \mathcal{A}_x - 2k_0 \frac{\partial \mathcal{A}_y}{\partial s} - k_0^2 \mathcal{A}_x = \frac{\mu_0 e^2 n \mathcal{A}_x}{m} \left(1 + \frac{2}{3} v_T^2 n^2 \right) + \frac{\omega_c k_0}{c}, \quad (48)$$

$$\square \mathcal{A}_y + 2k_0 \frac{\partial \mathcal{A}_x}{\partial s} - k_0^2 \mathcal{A}_y = \frac{\mu_0 e^2 n \mathcal{A}_y}{m} \left(1 + \frac{2}{3} v_T^2 n^2 \right). \quad (49)$$

$$\square a_s = \frac{\mu_0 e^2}{m} nV\Gamma. \quad (50)$$

of equations, which will be the starting point for the subsequent analysis.

Worthwhile to mention is that, instead of the Lorentz gauge adopted in the present paper, it is possible to use the Coulomb gauge, where $\partial_s a_s = 0$. In this case it is necessary to take Eq. (47) in the form

$$\nabla^2 \varphi = \partial_s^2 \varphi = \frac{e^2}{\epsilon_0 m c^2} n\Gamma, \quad (51)$$

while Eq. (50) transforms as

$$\square a_s - \partial_\tau \partial_s \varphi = -\partial_\tau \mathcal{F} = \frac{\mu_0 e^2}{m} nV\Gamma. \quad (52)$$

It is important to note that the exact hydrodynamic model derived here is invariant under Lorentz transformation. Detailed proof of this assertion is given in Appendix A.

IV. LINEAR STABILITY OF THE RELATIVISTIC HYDRODYNAMIC MODEL

Let us consider in what follows the simplest case of a cold electron beam, that is the limiting case, where $nv_T \rightarrow 0$. First of all, we note that the hydrodynamic equations (45) and (46) admit a stationary solution of the form $n = n_0 = \text{const}$ and $V = v_0 = \text{const}$. In addition, the stationary transverse components of the vector potential are given by the expressions

$$\mathcal{A}_{x0} = -\frac{\omega_c k_0 c}{\omega_p^2 + k_0^2 c^2}, \quad \mathcal{A}_{y0} = 0, \quad (53)$$

where ω_p is the plasma frequency expressed as

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m}. \quad (54)$$

Note also that in obtaining the stationary solutions above, the explicit assumption that the beam density n_0 is small has been made. Therefore, the longitudinal space-charge effects can be neglected (φ_0 , a_{s0} and \mathcal{F}_0 are all zero). Finally,

$$\Gamma_0 = \gamma_0 R_0, \quad \gamma_0 = \frac{1}{\sqrt{1 - v_0^2}}, \quad R_0 = \sqrt{1 + \mathcal{A}_{x0}^2}. \quad (55)$$

Instead of the full system of equations (45) - (50), we shall use

$$\frac{\partial}{\partial \tau}(n\Gamma) + \frac{\partial}{\partial s}(n\Gamma V) = 0, \quad (56)$$

$$\frac{\partial^2}{\partial \tau \partial s}(V\Gamma) + \frac{\partial^2 \Gamma}{\partial s^2} = \frac{e^2}{\epsilon_0 m c^2} n\Gamma, \quad (57)$$

coupled with the equations for the transverse vector potential (48) and (49). Equation (57) has been obtained by differentiating Eq. (46) with respect to the longitudinal coordinate s and making use of the Lorentz gauge condition (30) and the equation for the scalar potential (47).

Following the standard procedure, we take

$$n = n_0 + \epsilon n_1 + \dots, \quad V = v_0 + \epsilon v_1 + \dots, \quad (58)$$

$$\mathcal{A}_x = \mathcal{A}_{x0} + \epsilon \mathcal{A}_{x1} + \dots, \quad \mathcal{A}_y = \epsilon \mathcal{A}_{y1} + \dots, \quad (59)$$

and expand the quantities Γ , ΓV , $n\Gamma$ and $n\Gamma V$ to first order in the formal expansion parameter ϵ . Substituting all of the above into Eqs. (56), (57), (48) and (49), and retaining linear terms, we obtain

$$\begin{aligned} &(\partial_\tau + v_0 \partial_s) n_1 + n_0 \gamma_0^2 (v_0 \partial_\tau + \partial_s) v_1 \\ &+ \frac{n_0 \mathcal{A}_{x0}}{R_0^2} (\partial_\tau + v_0 \partial_s) \mathcal{A}_{x1} = 0, \end{aligned} \quad (60)$$

$$\begin{aligned} &\gamma_0^2 \partial_s (\partial_\tau + v_0 \partial_s) v_1 + \frac{\mathcal{A}_{x0}}{R_0^2} \partial_s (v_0 \partial_\tau + \partial_s) \mathcal{A}_{x1} \\ &= \frac{\omega_p^2}{c^2} \left(\frac{\mathcal{A}_{x0}}{R_0^2} \mathcal{A}_{x1} + v_0 \gamma_0^2 v_1 + \frac{n_1}{n_0} \right), \end{aligned} \quad (61)$$

$$\square \mathcal{A}_{x1} - 2k_0 \partial_s \mathcal{A}_{y1} - k_1^2 \mathcal{A}_{x1} = \frac{\omega_p^2 \mathcal{A}_{x0}}{c^2 n_0} n_1, \quad (62)$$

$$\square \mathcal{A}_{y1} + 2k_0 \partial_s \mathcal{A}_{x1} - k_1^2 \mathcal{A}_y = 0, \quad (63)$$

where

$$k_1^2 = k_0^2 + \frac{\omega_p^2}{c^2}. \quad (64)$$

It is convenient to perform a Lorentz transformation by introducing a new time and longitudinal coordinate variables according to the expressions

$$\theta = \gamma_0 (\tau - v_0 s), \quad z = \gamma_0 (s - v_0 \tau). \quad (65)$$

Derivatives transform according to

$$\partial_\tau = \gamma_0 (\partial_\theta - v_0 \partial_z), \quad \partial_s = \gamma_0 (-v_0 \partial_\theta + \partial_z). \quad (66)$$

Differential operators entering Eqs. (60) and (61) simplify considerably

$$\partial_\tau + v_0 \partial_s = \frac{1}{\gamma_0} \partial_\theta, \quad v_0 \partial_\tau + \partial_s = \frac{1}{\gamma_0} \partial_z, \quad (67)$$

while the d'Alembert operator \square is invariant in the inertial frame moving in the longitudinal direction at speed v_0 . Thus, in the new Lorentz coordinate system the linearized equations simplify considerably

$$\partial_\theta n_1 + n_0 \gamma_0^2 \partial_z v_1 + \frac{n_0 \mathcal{A}_{x0}}{R_0^2} \partial_\theta \mathcal{A}_{x1} = 0, \quad (68)$$

$$\begin{aligned} &\gamma_0^2 \partial_\theta (-v_0 \partial_\theta + \partial_z) v_1 + \frac{\mathcal{A}_{x0}}{R_0^2} \partial_z (-v_0 \partial_\theta + \partial_z) \mathcal{A}_{x1} \\ &= \frac{\omega_p^2}{c^2} \left(\frac{\mathcal{A}_{x0}}{R_0^2} \mathcal{A}_{x1} + v_0 \gamma_0^2 v_1 + \frac{n_1}{n_0} \right), \end{aligned} \quad (69)$$

$$(\square - k_1^2) \mathcal{A}_{x1} - 2k_0 \gamma_0 (-v_0 \partial_\theta + \partial_z) \mathcal{A}_{y1} = \frac{\omega_p^2 \mathcal{A}_{x0}}{c^2 n_0} n_1, \quad (70)$$

$$(\square - k_1^2) \mathcal{A}_{y1} + 2k_0 \gamma_0 (-v_0 \partial_\theta + \partial_z) \mathcal{A}_{x1} = 0. \quad (71)$$

Manipulating the first two Eqs. (68) and (69), we can eliminate the linear velocity v_1 and obtain a single equation relating the linear density n_1 and the horizontal component of the vector potential. We have

$$(-v_0 \partial_\theta + \partial_z) \left[\frac{\mathcal{A}_{x0}}{R_0^2} \left(\square - \frac{\omega_p^2}{c^2} \right) \mathcal{A}_{x1} - \left(\partial_\theta^2 + \frac{\omega_p^2}{c^2} \right) \frac{n_1}{n_0} \right] = 0. \quad (72)$$

It is now a simple matter to obtain a single equation for the horizontal component of the vector potential. The result is

$$(-v_0\partial_\theta + \partial_z)\hat{\mathfrak{N}}\mathcal{A}_{x1} = 0, \quad (73)$$

where the operator $\hat{\mathfrak{N}}$ is given by

$$\begin{aligned} \hat{\mathfrak{N}} = & \left(\partial_\theta^2 + \frac{\omega_p^2}{c^2} \right) (\square - k_1^2)^2 + 4k_0^2\gamma_0^2 \left(\partial_\theta^2 + \frac{\omega_p^2}{c^2} \right) \\ & \times (-v_0\partial_\theta + \partial_z)^2 - \frac{\omega_p^2\mathcal{A}_{x0}^2}{c^2R_0^2} (\square - k_1^2) \left(\square - \frac{\omega_p^2}{c^2} \right). \end{aligned} \quad (74)$$

Note that, the left-hand-side of Eq. (73) represents a product of two operators. In the laboratory frame the action of the first operator $(-v_0\partial_\theta + \partial_z)$ simply implies that the linear solution \mathcal{A}_{x1} to our fluid dynamic model does not depend on the longitudinal variable s and therefore is an arbitrary function of time to this end. This observation is consistent with Eqs. (60) and (61), provided the condition

$$\frac{\mathcal{A}_{x0}}{R_0^2}\mathcal{A}_{x1} + v_0\gamma_0^2v_1 + \frac{n_1}{n_0} = 0, \quad (75)$$

holds for the time dependent first order quantities. Note also that, if the velocity $v_1(\tau)$ is chosen arbitrarily (but dependent on time τ only), then $n_1(\tau)$ and $\mathcal{A}_{x1}(\tau)$ can be determined uniquely from Eqs. (62) and (75). Namely, expressing n_1 in terms of \mathcal{A}_{x1} and v_1 from Eq. (75) and substituting the result into Eq. (62), we obtain

$$\partial_\tau^2\mathcal{A}_{x1} + \omega_h^2\mathcal{A}_{x1} = \frac{\omega_p^2}{c^2}\mathcal{A}_{x0}v_0\gamma_0^2v_1, \quad (76)$$

where

$$\omega_h^2 = k_0^2 + \frac{\omega_p^2}{c^2R_0^2}, \quad (77)$$

is the characteristic frequency of harmonic oscillations.

In the next Section, we analyze in detail the Fourier spectrum of the operator $\hat{\mathfrak{N}}$.

V. NUMERICAL RESULTS FOR THE SMALL-SIGNAL GAIN

Let us rewrite the operator defined in Eq. (74) in the laboratory frame as follows

$$\begin{aligned} \hat{\mathfrak{N}} = & \left[\gamma_0^2(\partial_\tau + v_0\partial_s)^2 + \frac{\omega_p^2}{c^2} \right] \left[(\square - k_1^2)^2 + 4k_0^2\partial_s^2 \right] \\ & - \frac{\omega_p^2\mathcal{A}_{x0}^2}{c^2R_0^2} (\square - k_1^2) \left(\square - \frac{\omega_p^2}{c^2} \right). \end{aligned} \quad (78)$$

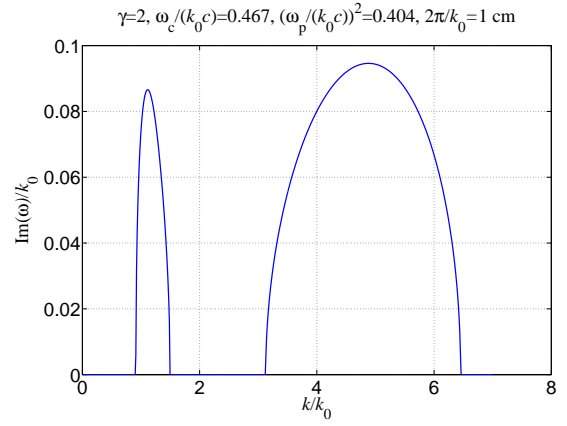


FIG. 1. Free electron laser instability growth rate as a function of the normalized radiation wave number.

Next, we seek a solution to the equation $\hat{\mathfrak{N}}\mathcal{A}_{x1} = 0$ in the standard form

$$\mathcal{A}_{x1} = \sum_n \int_{-\infty}^{\infty} dk \mathcal{A}(k) \exp[iks - i\omega_n(k)\tau], \quad (79)$$

where $\omega_n(k)$ are all possible solutions of the *dispersion equation*

$$\begin{aligned} & \left[\frac{\omega_p^2}{c^2} - \gamma_0^2(kv_0 - \omega)^2 \right] \\ & \times \left[\omega^2 - (k + k_0)^2 - \frac{\omega_p^2}{c^2} \right] \left[\omega^2 - (k - k_0)^2 - \frac{\omega_p^2}{c^2} \right] \\ & = \frac{\omega_p^2\mathcal{A}_{x0}^2}{c^2R_0^2} (\omega^2 - k^2 - k_1^2) \left(\omega^2 - k^2 - \frac{\omega_p^2}{c^2} \right). \end{aligned} \quad (80)$$

In order to assess the predictions of the hydrodynamic model on a qualitative and quantitative level, Eq. (80) has been solved numerically and the results are presented in Figures 1 – 5. Note, that in the notation adopted here the frequency $[\omega] = m^{-1}$ is, in fact, the radiation wave number.

At low γ_0 values (Figures 1 and 2) the observed behavior of the free electron laser instability growth rate is qualitatively similar to that reported earlier¹³. Note, however, that a hydrodynamic approach is adopted here, whereas a kinetic description has been used by Davidson and Uhm¹³. In addition there is a gamma factor difference in the definitions of ω_p and ω_c in the paper by Davidson and Uhm¹³ and here.

At higher γ_0 and undulator magnetic field strength values (see Figure 3) the instability bandwidth increases and the location of its peak shifts towards shorter radiation wavelengths.

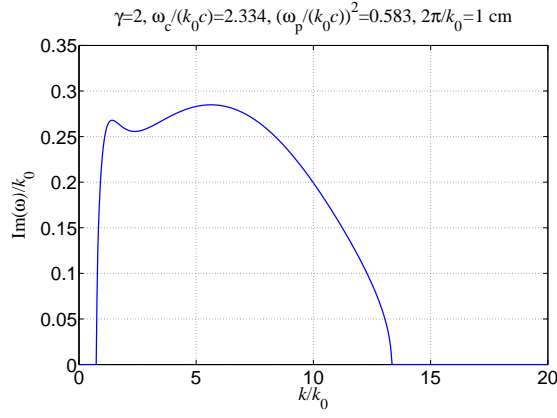


FIG. 2. The same as in Fig. 1 but at higher magnetic field and beam density values.

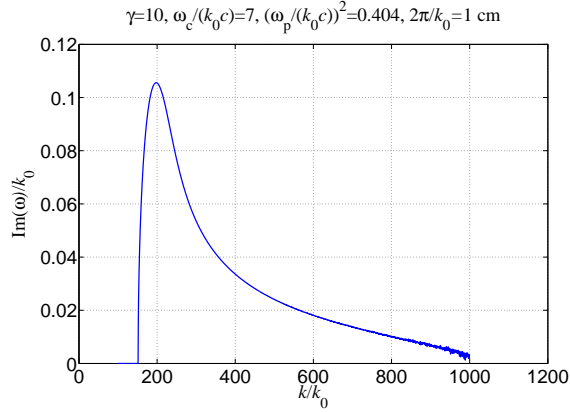


FIG. 3. The same as Fig. 1 but at higher beam energy and magnetic field strength.

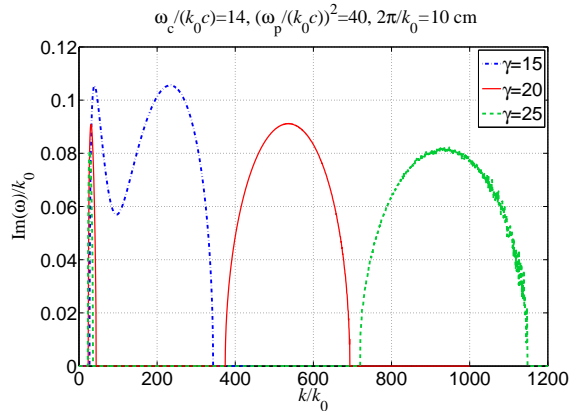


FIG. 4. Dependence of the instability growth rate on beam energy for fixed magnetic field, beam density and undulator period values.

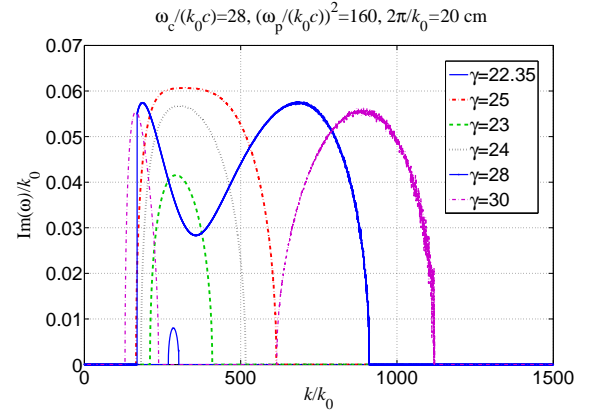


FIG. 5. The same as in Fig. 4 but the undulator period length has been increased by a factor of 2.

Figures 4 and 5 show the effect of beam energy on the instability at two different undulator period lengths. As can be seen at lower beam energies a single instability band exists, whereas at higher beam energies two separate bands are generated. In addition (see Figure 5), there is a threshold beam energy below which no instability exists. Increasing beam energy above its threshold value an increase of both the bandwidth (range of k -values) and the peak value of ω as a function of k can be observed.

VI. CONCLUDING REMARKS

Starting from first principle an exact relativistic hydrodynamic closure of equations describing the dynamics of an electron beam propagating axially in a static magnetic undulator field has been obtained. It has been shown that the hydrodynamic equations are fully equivalent to the Vlasov-Maxwell system for a special type of solutions of the Vlasov equation. Moreover, as expected, the warm (in general) fluid dynamic equations derived in the present paper are invariant under Lorentz transformation.

Another interesting feature of our hydrodynamic picture is the underlying pressure law. The latter together with the ponderomotive potential (usually proportional to the transverse vector potential squared) represents an effective enthalpy of the system [see Eq. (28)]. Noteworthy to mention is also that in the non relativistic limit our system of hydrodynamic equations reduces to the well-known picture with triple adiabatic pressure law^{20,21}.

As a direct application of the theory developed here, the linear stability of the exact hydrodynamic closure has been studied. It has been shown that all basic features of the small signal gain can be derived from the fluid dynamic description developed in the present paper. Satisfactory agreement with previously reported results has been also found.

A possible extension of the approach initiated here could be a numerical modeling of the hydrodynamic equations, as well as analysis of nonlinear effects and possible formation of solitary wave patterns and coherent structures. All of the above we plan to perform in the near future.

Appendix A: Lorentz Invariance of the Hydrodynamic Model

Let us first represent the quantity Γ defined by Eq. (28) as

$$\Gamma = \gamma \Sigma, \quad (\text{A1})$$

where

$$\gamma = \frac{1}{\sqrt{1 - V^2}}, \quad \Sigma = \sqrt{\frac{1 + a^2}{1 - 2v_T^2 n^2}}. \quad (\text{A2})$$

In order to prove the relativistic invariance of our basic system of equations (45) - (50), we recall the Lorentz velocity addition law

$$V = \frac{V' + v_0}{1 + v_0 V'}, \quad \gamma = \gamma_0 \gamma' (1 + v_0 V'), \quad (\text{A3})$$

following from the Lorentz transformation (65). Here, with prime we denote the value of the corresponding quantity in a coordinate system moving in the longitudinal direction s with velocity v_0 . In addition, the scalar potential φ and the longitudinal component of the vector potential a_s transform according to the expressions

$$\varphi' = \gamma_0(\varphi - v_0 a_s), \quad a'_s = \gamma_0(a_s - v_0 \varphi), \quad (\text{A4})$$

Using expressions (66) for the corresponding derivatives, we obtain

$$\begin{aligned} & \gamma_0^2 (\partial_\theta - v_0 \partial_z) [n \Gamma' (1 + v_0 V')] \\ & + \gamma_0^2 (-v_0 \partial_\theta + \partial_z) [n \Gamma' (V' + v_0)] = 0. \end{aligned} \quad (\text{A5})$$

Simple rearrangement of terms in the above equation leads to Eq. (45) in the new coordinate system. In a similar way it can be verified that the left-hand-side of the momentum balance equation (46) does not change in the moving system. Since the longitudinal force \mathcal{F} is an obvious Lorentz invariant due to the transformation law of the electromagnetic potentials (A4), Eq. (46) also remains unchanged in the new coordinate system.

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